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MATHEMATICS

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TRIGONOMETRIC EQUATION (PHASE–II)

THEORY AND EXERCISE BOOKLET

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JEE Syllabus :

General solution of trigonometric equations.

A. SOLUTION OF TRIGONOMETRIC EQUATIONS

A solution of trigonometric equation is the value of the unknown angle that satisfies the equation.

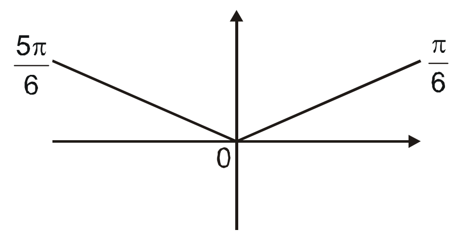
e.g. if $\sin\theta = \dfrac{1}{\sqrt{2}}$ $\quad\Rightarrow\quad$ $\theta = \dfrac{\pi}{4}, \dfrac{3\pi}{4}, \dfrac{9\pi}{4}, \dfrac{11\pi}{4}, ...$

Thus, the trigonometric equation may have infinite number of solutions (because of their periodic nature) and can be classified as : **(1)** Principal solution **(2)** General solution.

(1) Principal solutions : The solutions of trigonometric equation which lie in the interval $[0, 2\pi)$ are called **principal solutions.**

Ex.1 Find the Principal solutions of the equation $\sin x = \dfrac{1}{2}$.

Sol. $\because$ $\sin x = \dfrac{1}{2}$ $\because$ there exists two values



i.e. $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$ which lie in $[0, 2\pi)$ and whose sine is $\dfrac{1}{2}$

$\therefore$ Principal solutions of the equation $\sin x = \dfrac{1}{2}$ are $\dfrac{\pi}{6}, \dfrac{5\pi}{6}$

(2) General solution : The expression involving an integer 'n' which gives all solutions of a trigonometric equation is called **General solution.**

(a) If $\sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha$ where $\alpha \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$, $n \in I$.

(b) If $\cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha$ where $\alpha \in [0, \pi]$, $n \in I$.

(c) If $\tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha$ where $\alpha \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$, $n \in I$.

(d) If $\sin^2\theta = \sin^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

(e) $\cos^2\theta = \cos^2\alpha \Rightarrow \theta = n\pi \pm \alpha$.

(f) $\tan^2\theta = \tan^2\alpha \Rightarrow \theta = n\pi \pm \alpha$. [Note : α is called the principal angle]

Ex.2 Solve $\sec 2\theta = -\dfrac{2}{\sqrt{3}}$

Sol. $\sec 2\theta = -\dfrac{2}{\sqrt{3}} \Rightarrow \cos 2\theta = -\dfrac{\sqrt{3}}{2} \Rightarrow \cos 2\theta = \cos\dfrac{5\pi}{6} \Rightarrow 2\theta = 2n\pi \pm \dfrac{5\pi}{6}, n \in I \Rightarrow \theta = n\pi \pm \dfrac{5\pi}{12}, n \in I$

Ex.3 Solve $\tan\theta = 2$

Sol. $\because$ $\tan\theta = 2$ ....(i). Let $2 \tan\alpha$ $\Rightarrow$ $\tan\theta = \tan\alpha$ $\Rightarrow \theta = n\pi + \alpha$, where $\alpha = \tan^{-1}(2)$, $n \in I$

Ex.4 Solve $\cos^2\theta = \frac{1}{2}$

Sol. $\because$ $\cos^2\theta = \frac{1}{2} \Rightarrow$ $\cos^2\theta = \left(\frac{1}{\sqrt{2}}\right)^2$ $\Rightarrow$ $\cos^2\theta = \cos^2\frac{\pi}{4}$ $\Rightarrow$ $\theta + n\pi \pm \frac{\pi}{4}, n \in I$

Ex.5 Solve $4\tan^2\theta = 3\sec^2\theta$

Sol. $\because$ $4\tan^2\theta = 3\sec^2\theta$ ...(i) For equation (i) to be defined $\theta \neq (2n+1)\frac{\pi}{2}, n \in I$

$\because$ equation (i) can be written as : $\frac{4\sin^2\theta}{\cos^2\theta} = \frac{3}{\cos^2\theta}$ $\because$ $\theta \neq (2n+1)\frac{\pi}{2}, n \in I$

$\Rightarrow$ $4\sin^2\theta = 3$ $\therefore$ $\cos^2\theta \neq 0$

$\Rightarrow$ $\sin^2\theta = \left(\frac{\sqrt{3}}{2}\right)^2$ $\Rightarrow$ $\sin^2\theta = \sin^2\frac{\pi}{3}$ $\Rightarrow$ $\theta + n\pi \pm \frac{\pi}{3}, n \in I$

B. SOLUTIONS OF EQUATIONS BY FACTORISING

Ex.6 Solve $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

Sol. $\because$ $(2\sin x - \cos x)(1 + \cos x) = \sin^2 x$

$\Rightarrow$ $(2\sin x - \cos x)(1 + \cos x) - (1 - \cos x)(1 + \cos x) = 0$ $\Rightarrow$ $(1 + \cos x)(2\sin x - 1) = 0$

$\Rightarrow$ $1 + \cos x = 0$ or $2\sin x - 1 = 0$ $\Rightarrow$ $\cos x = -1$ or $\sin x = \frac{1}{2}$

$\Rightarrow$ $x = (2n+1)\pi, n \in I$ or $\sin x = \sin\frac{\pi}{6}, n \in I$ $\Rightarrow$ $x + n\pi(-1)^n\frac{\pi}{6}, n \in I$

$\therefore$ Solution of given equation is $(2n+1)\pi, n \in I$ or $n\pi + (-1)^n\frac{\pi}{6}, n \in I$

Ex.7 Solve the equation $\sin^3 x\cos x - \sin x\cos^3 x = \frac{1}{4}$.

Sol. The equation can be written as $4\sin x\cos x(\sin^2 x - \cos^2 x) = 1$,

$\Rightarrow -2\sin 2x\cos 2x = -\sin 4x = 1$ $\Rightarrow$ $x = -\frac{\pi}{8} + k\,.\,\frac{\pi}{2}$ $(k = 0, \pm 1, \pm 2, ...)$

Ex.8 Solve the equation $\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x} - \frac{1}{\tan^2 x} - \frac{1}{\cot^2 x} - \frac{1}{\sec^2 x} - \frac{1}{\csc^2 x} = -3$

Sol. Using the well-known trigonometric formulas, write the equation in the following way :

$\csc^2 x - \sec^2 x - \cot^2 x - \tan^2 x - \cos^2 x - \sin^2 x = -3$ ...(1)

Since $\csc^2 x = 1 + \cot^2 x$ and $\sec^2 x = 1 + \tan^2 x$, the above equation is reduced to the form $\tan^2 x = 1$

$\Rightarrow$ $x = \frac{\pi}{4} + k\frac{\pi}{2}$.

Ex.9 Find the general solution of the equation $\dfrac{1+\sin x+\sin^2 x+\sin^3 x+......+\sin^n x+....\infty}{1-\sin x+\sin^2 x-\sin^3 x+..(-1)^n \sin^n x+...\infty} = \dfrac{4}{1+\tan^2 x}$

where $x \neq k\pi + \dfrac{\pi}{2}$, $k \in I$.

Sol. N^r of LHS = $\dfrac{1}{1-\sin x}$; D^r of LHS = $\dfrac{1}{1+\sin x}$

hence $\dfrac{1+\sin x}{1-\sin x} = \dfrac{4}{\sec^2 x} = 4\cos^2 x = 4(1-\sin x)(1+\sin x)$

hence $4(1-\sin x)^2 = 1 \Rightarrow (1-\sin x)^2 = \dfrac{1}{4} \Rightarrow (1-\sin x) = \dfrac{1}{2}$ or $-\dfrac{1}{2}$

$\therefore \sin x = \dfrac{1}{2}$ or $\sin x = \dfrac{3}{2}$ (rejected) $\therefore \sin x = \sin\dfrac{\pi}{6} \Rightarrow x = n\pi + (-1)^n\dfrac{\pi}{6}$, $n \in I$

Ex.10 Find the general solution of the equation $\sin^3 x(1 + \cot x) + \cos^3 x(1 + \tan x) = \cos 2x$.

Sol. $\sin^2 x(\cos x + \sin x) + \cos^2 x(\cos x + \sin x) = \cos 2x$

$(\cos x + \sin x)(\cos^2 x + \sin^2 x) = (\cos x + \sin x)(\cos x - \sin x)$

$\therefore (\cos x + \sin x)[\cos x - \sin x - 1)] = 0$

$\therefore$ either $\cos x + \sin x = 0$ ....(1) or $\cos x - \sin x = 1$ ....(2)

from (1) $\tan x = -1$ or $1 - \sin 2x = 1 \Rightarrow \sin 2x = 0$

If $\tan x = -1 = \tan\left(-\dfrac{\pi}{4}\right)$ $\therefore$ $x = x = n\pi - \dfrac{\pi}{4}$, $n \in I$

If $\sin 2x = 0 \Rightarrow 2x = n\pi \Rightarrow x = \dfrac{n\pi}{2}$ this is to be rejected because of the tan x or cot x will not be defined so $x = \left(n\pi - \dfrac{\pi}{4}\right)$, $n \in I$

Ex.11 Find the solutions of the equation, $\log_{\sqrt{2}\sin x}(1+\cos x) = 2$ in the interval $x \in [0, 2\pi]$.

Sol. $2\sin^2 x = 1 + \cos x$; $2\cos^2 x + \cos x - 1 = 0$

$\Rightarrow \cos x = \dfrac{1}{2}$ or -1 $\Rightarrow x = \dfrac{\pi}{3}, \pi, \dfrac{5\pi}{3}$ but $x = \pi$ and $\dfrac{5\pi}{3}$ are rejected $\Rightarrow x = \dfrac{\pi}{3}$

Ex.12 Find the general solution of the trigonometric equations ,

(a) $\sec\left(\dfrac{\pi}{4}+x\right) + \sec\left(\dfrac{\pi}{4}-x\right) = 2\sqrt{2}$, **(b)** $\begin{vmatrix} 1 & \cos x & \cos^2 x \\ 1 & \cos\alpha & \cos^2\alpha \\ 1 & -\cos\alpha & -\cos^2\alpha \end{vmatrix} = 0$ where $\alpha \in \left(0, \dfrac{\pi}{2}\right)$

What happens if $\alpha = \pi/2$.

Sol. **(a)** $\dfrac{\sqrt{2}}{\cos x - \sin x} + \dfrac{\sqrt{2}}{\cos x + \sin x} = 2\sqrt{2}$ or $\dfrac{2\cos x}{\cos 2x} = 2 \Rightarrow \cos 2x = \cos x$

Hence $2x = 2n\pi \pm x$, positive sign , $x = 2n\pi$; negative sign , $x = \dfrac{2n\pi}{3}$

Hence the general solution is $x = \dfrac{2n\pi}{3}$, $n \in I$ as $2n\pi$ is a subset of $\dfrac{2n\pi}{3}$

(b) $\begin{vmatrix} 0 & \cos x - \cos\alpha & \cos^2 x - \cos^2\alpha \\ 0 & 2\cos\alpha & 0 \\ 1 & -\cos\alpha & \cos^2\alpha \end{vmatrix} = 0$

$\Rightarrow 2\cos\alpha\ (\cos^2 x - \cos^2\alpha) = 0 \Rightarrow \cos^2 x = \cos^2\alpha \quad (\cos\alpha \neq 0) \quad \Rightarrow \quad x = n\pi \pm \alpha$

If $\alpha = \dfrac{\pi}{2}$ the equation becomes an identity and hence true $\forall\ x \in R$.

C. SOLUTIONS OF EQUATIONS REDUCIBLE TO QUADRATIC EQUATIONS

Ex.13 Solve the equation $\sin^2 x(\tan x + 1) = 3\sin x(\cos x - \sin x) + 3$.

Sol. The given equation makes no sense when $\cos x = 0$; therefore we can suppose that $\cos x \neq 0$. Noting that the right-hand member of the equation is equal to $3\sin x\cos x + 3\cos^2 x$, and dividing both members by $\cos^2 x$, we obtain $\tan^2 x(\tan x + 1) = 3(\tan x + 1)$,

$\Rightarrow (\tan^2 x - 3)(\tan x + 1) = 0 \quad \Rightarrow x_1 = -\dfrac{\pi}{4} + k\pi,\ x_2 = \dfrac{\pi}{3} + k\pi,\ x_3 = -\dfrac{\pi}{3} + k\pi.$

Ex.14 Find the general solution set of the equation $\log_{\tan x}(2 + 4\cos^2 x) = 2$.

Sol. $2 + 4\cos^2 x = \tan^2 x \Rightarrow 3 + 4\cos^2 x = \sec^2 x \Rightarrow 4\cos^4 x + 3\cos^2 x - 1 = 0$

let $\cos^2 x = t \Rightarrow 4t^2 + 3t - 1 = 0 \Rightarrow (4t - 1)(t + 1) = 0 \Rightarrow t = 1/4$ or $t = -1$

$\Rightarrow \cos^2 x = \dfrac{1}{4}$ or $\cos^2 x = -1$ (not possible) $\Rightarrow \cos^2 x = \cos^2\dfrac{\pi}{3} \Rightarrow x = n\pi + \dfrac{\pi}{3},\ n \in I$

Ex.15 The equation $\cos^2 x - \sin x + a = 0$ has roots when $x \in (0, \pi/2)$ find 'a'.

Sol. $1 - \sin^2 x - \sin x + a = 0 \Rightarrow \sin^2 x + \sin x - (a + 1) = 0$ (let $\sin x = t$)

$\therefore\ t^2 + t - (a + 1) = 0,\ t \in (0, 1) \Rightarrow t = \dfrac{-1 \pm \sqrt{1 + 4(a+1)}}{2} \Rightarrow t = \dfrac{-1 \pm \sqrt{4a + 5}}{2}$ (reject – ve sign)

$\therefore\ t = \dfrac{-1 + \sqrt{4a+5}}{2}$ now $0 < \dfrac{-1 + \sqrt{4a+5}}{2} < 1$

$\Rightarrow 0 < -1 + \sqrt{4a+5} < 2$ or $1 < \sqrt{4a+5} < 3$

$\Rightarrow 1 < 4a + 5 < 9 \Rightarrow -4 < 4a < 4 \Rightarrow -1 < a < 1 \Rightarrow a \in (-1, 1)$

Ex.16 Solve the equation $\cot^2 x = \frac{1+\sin x}{1+\cos x}$

Sol. The given equation only makes sense for $x \neq k\pi$. For these values of x it can be rewritten in the form $\cos^3 x + \cos^2 x = \sin^3 x + \sin^2 x$.

Transferring all terms to the left-hand side of the equation and factoring it we get

$(\cos x - \sin x)(\sin^2 x + \cos^2 x + \sin x \cos x + \sin x + \cos x) = 0.$

There are two possible cases here which are considered below.

(a) $\sin x - \cos x = 0$, then $x_1 = \frac{\pi}{4} + k\pi$; ......(1)

(b) $\sin^2 x = \cos^2 x + \sin x \cos x + \sin x + \cos x = 0.$ ......(2)

Equation (2) has the solutions $x_2 = -\frac{\pi}{2} + 2k\pi$ ......(3) and $x_3 = (2k + 1)\pi$. ......(4)

But the values of x determined by formula (4) are not roots of the original equations, since the original equation is only considered for $x \neq k\pi$. Consequently, the equation has the roots defined by formulas (1) and (3).

Ex.17 Solve the equation $2 - (7 + \sin 2x)\sin^2 x + (7 + \sin 2x)\sin^4 x = 0$.

Sol. The left member of the equation being equal to

$2 - (7 + \sin 2x)(\sin^2 x - \sin^4 x) = 2 - (7 + \sin 2x)\sin^2 x \,.\, \cos^2 x = 2 - (7 + \sin 2x)\frac{1}{4}\sin^2 2x,$

we can put $t = \sin 2x$ and rewrite the equation in the form $t^3 + 7t^2 - 8 = 0$ ......(1)

It is readily see that equation (1) has the roots $t_1 = 1$. The other two roots are found form the equation $t^2 + 8x + 8 = 0$ ......(2)

Solving this equation we find $t = -4 + 2\sqrt{2}$ and $t = -4 - 2\sqrt{2}$.

These roots should be discarded because they are greater than unity in their absolute values. Consequently, the roots of the original equation coincide with the roots of the equation $\sin 2x = 1$.

$x = \pi/4 + k\pi$

Ex.18 Solve the equation $\sin^8 x + \cos^8 x = \frac{17}{32}$

Sol. Using the identity $(\sin^2 x + \cos^2 x)^2 = 1$ we get $\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$,

whence $\sin^8 x + \cos^8 x = \left(1 - \frac{1}{2}\sin^2 2x\right)^2 - \frac{1}{8}\sin^4 2x = \frac{17}{32}$

$\Rightarrow \quad 1 - \sin^2 2x + \frac{1}{8}\sin^4 2x = \frac{17}{32} \quad \Rightarrow \quad \sin^4 2x - 8\sin^2 2x + \frac{15}{4} = 0.$

Solving we get $\sin^2 2x = 4 \pm \frac{7}{2}$, $\sin^2 2x = \frac{1}{2}$, $2x = \frac{\pi}{4} + k\frac{\pi}{2}$; whence $x = \frac{2k+1}{8}\pi$.

Ex.19 Find all solutions of the equation $(\tan^2 x - 1)^{-1} = 1 + \cos 2x$, which satisfy the inequality $2^{x+1} - 8 > 0$

Sol. Let us reduce the initial trigonometric equation to the form $(1 + \cos 2x)\left(1+\frac{1}{2\cos 2x}\right) = 0$.

The following values of x are solutions of this equation $x = -\frac{\pi}{2} + \pi n, \quad x = \pm\frac{\pi}{3} + \pi k, \quad n, \quad k \in Z$.

By the hypothesis, we must choose those values of x which satisfy the inequalities

$2^{x+1} - 8 > 0, \quad \cos x \neq 0$. The values we need are $x = \pm\frac{\pi}{3} + \pi n, n \in N$

Ex.20 For what a is the equation $\sin^2 x - \sin x \cos x - 2\cos^2 x = a$ solvable ? Find the solutions.

Sol. Multiplying the right member of the equation by $\sin^2 x + \cos^2 x = 1$ we reduce it to the form

$(1 - \alpha)\sin^2 x - \sin x \cos x - (a + 2)\cos^2 x = 0$. ....(1)

First let us assume that $a \neq 1$. Then form (1) is follows that $\cos x \neq 0$, since otherwise we have $\sin x = \cos x = 0$ which is impossible. Dividing both members of (1) by $\cos^2 x$ and putting $\tan x = t$ we get the equation $(1 - a)t^2 - t - (a + 2) = 0$. ....(2)

Equation (1) is solvable if and only if the roots of equation (2) are real, i.e. if its discriminant is non-negative $D = -4a^2 - 4a + 9 \geq 0$. ....(3)

Solving inequality (3) we find $-\frac{\sqrt{10}+1}{2} \leq a \leq \frac{\sqrt{10}-1}{2}$ ....(4)

Let t_1 and t_2 be the roots of equation (2). Then the corresponding solutions of equation (1) have the form $x_1 = \text{arc}\tan t_1 + k\pi$, $x_2 = \text{arc}\tan t_2 + k\pi$,

Now let us consider the case $a = 1$.

In this case equation (1) is written in the form $\cos x(\sin x + 3\cos x) = 0$

and has the following solutions : $x_1 = \frac{\pi}{2} + k\pi, \quad x_2 = -\text{arc}\tan 3 + k\pi$.

Ex.21 Determine all the values of a for which the equation $\sin^4 x - 2\cos^2 x + a^2 = 0$ is solvable. Find the solutions.

Sol. Applying the formula $\sin^4 x = \left(\frac{1-\cos 2x}{2}\right)^2$, $\cos^2 x = \frac{1+\cos 2x}{2}$ and putting $\cos 2x = t$

we rewrite the given equation in the form $t^2 - 6t + 4a^2 - 3 = 0$ ....(1)

The original equation has solutions for a given value of a if and only if, for his value of a, the roots t_1 and t_2 of the equation (1) are real and at least one of these roots does not exceed unity in its absolute value. Solving equation (1), we find $t_1 = 3 - 2\sqrt{3-a^2}$, $\quad t_2 = 3 + 2\sqrt{3-a^2}$.

Hence the rotos of equation (1) are real if $|a| \leq \sqrt{3}$ ....(2)

If condition (2) is fulfilled, then $t_2 > 1$ and, therefore, this root can be dicarded. Thus, the problem is reduced to finding the values of a satisfying condition (2), for which $|t_1| \leq 1$, i.e.,

$-1 \leq 3 - 2\sqrt{3-a^2} \leq 1$. ....(3)

From (3) we find $-4 \leq -2\sqrt{3-a^2} \leq -2, \quad \Rightarrow \quad 2 \geq \sqrt{3-a^2} \geq 1$. ....(4)

Since the inequality $2 \geq \sqrt{3-a^2}$ is fulfilled for $|a| \leq \sqrt{3}$, the system of inequalities (4) is reduced to the inequality $\sqrt{3-a^2} \geq 1, \quad \Rightarrow |a| \leq \sqrt{2}$.

Thus, the original equation is solvable if $|a| \leq \sqrt{2}$, and its solutions are

$x = \pm\frac{1}{2}\text{arc}\cos(3 - 2\sqrt{3-a^2}) + k\pi$.

D. SOLVING EQUATIONS BY INTRODUCING AN AUXILIARY ARGUMENT

Ex. 22 Solve sin x + cos x = $\sqrt{2}$

Sol. $\because$ sin x + cos x = $\sqrt{2}$ ....(i) Here a = 1, b = 1

$\therefore$ divide both sides of equation (i) by $\sqrt{2}$, we get sin x . $\frac{1}{\sqrt{2}}$ + cos x . $\frac{1}{\sqrt{2}}$ = 1

$\Rightarrow$ sin x . sin $\frac{\pi}{4}$ + cos x . cos $\frac{\pi}{4}$ = 1 $\Rightarrow$ cos $\left(x - \frac{\pi}{4}\right)$ = 1 $\Rightarrow$ x − $\frac{\pi}{4}$ = 2nπ, n $\in$ I

$\Rightarrow$ x = 2nπ + $\frac{\pi}{4}$, n $\in$ I $\therefore$ Solution of given equation is 2nπ + $\frac{\pi}{4}$, n $\in$ I

Ex.23 Solve the equation cos 7x − sin 5x = $\sqrt{3}$ (cos 5x − sin 7x).

Sol. Rewrite the equation in the form $\frac{1}{2}$ cos 7x + $\frac{\sqrt{3}}{2}$ sin 7x = $\frac{\sqrt{3}}{2}$ cos 5x + $\frac{1}{2}$ sin 5x

or sin $\frac{\pi}{6}$ cos 7x + cos $\frac{\pi}{6}$ sin 7x = sin $\frac{\pi}{3}$ cos 5x + cos $\frac{\pi}{3}$ sin 5x, i.e. sin $\left(\frac{\pi}{6} + 7x\right) = \sin\left(\frac{\pi}{3} + 5x\right)$.

But sin α = sin β if and only if either $\alpha - \beta$ = 2kπ or $\alpha + \beta$ = (2m + 1) π (k, m = 0, ± 1, ± 2, ...).

Hence $\frac{\pi}{6} + 7x - \frac{\pi}{3}$ − 5x = 2kπ or $\frac{\pi}{6} + 7x - \frac{\pi}{3}$ − 5x = (2m + 1) π.

Thus, the roots for the equation are $\left.\begin{aligned} x &= \frac{\pi}{12}(12k+1), \\ x &= \frac{\pi}{24}(4m+1) \end{aligned}\right\}$ (k, m = 0, ± 1, ± 2,.....).

Ex.24 Solve the equation $\frac{a\sin x + b}{b\cos x + a} = \frac{a\cos x + b}{b\sin x + a}$ ($a^2 \neq 2b^2$)

Sol. Noting that (b cos x + a) (b sin x + a) ≠ 0 (otherwise the equation has no sense), we discard the denominators and get ab $\sin^2$ x + (a^2 + b^2) sin x + ab = ab $\cos^2$ x + (a^2 + b^2) cos x + ab,
whence (a^2 + b^2) (sin x − cos x) − ab ($\sin^2$ x − $\cos^2$ x) = 0.
Therefore, the original equation is reduced to the following two equations :

1°. sin x = cos x, whence x = $\frac{\pi}{4}$ + kπ and 2° . sin x + cos x = $\frac{a^2+b^2}{ab}$.

But the latter equation has no solutions because $\frac{a^2+b^2}{|ab|} \geq 2$.

whereas |sin x + cos x | = $\sqrt{2}\left|\sin x.\frac{1}{\sqrt{2}} + \cos x.\frac{1}{\sqrt{2}}\right| = \sqrt{2}\left|\sin\left(x + \frac{\pi}{4}\right)\right| \leq \sqrt{2}$ $\Rightarrow$ x = $\frac{\pi}{4}$ + kπ

Ex.25 Solve the equation $2\sin 17x + \sqrt{3}\cos 5x + \sin 5x = 0$

Sol. Dividing both sides of the equation by 2, we reduce it to the form $\sin 17x + \sin\left(5x + \frac{\pi}{3}\right) = 0$,

whence we obtain $2\sin\left(11x + \frac{\pi}{6}\right)\cos\left(6x - \frac{\pi}{6}\right) = 0 \Rightarrow x_1 = -\frac{\pi}{66} + \frac{k\pi}{11},\ x_2 = \frac{\pi}{36} + \frac{(2k+1)\pi}{12}$

Ex.26 Solve the equation $\sin^3 x + \cos^3 x = 1 - \frac{1}{2}\sin 2x$.

Sol. Using the formula for the sum of cubes of two members we transform the left-hand side of the equation in the following way : $(\sin x + \cos x)(1 - \sin x\cos x) = \left(1 - \frac{1}{2}\sin 2x\right)(\sin x + \cos x)$.

Hence, the original equation takes the form $\left(1 - \frac{1}{2}\sin 2x\right)(\sin x + \cos x - 1) = 0$.

The expression in the first brackets is different from zero for all x. Therefore it is sufficient to consider the equation $\sin x + \cos x - 1 = 0$. The latter is reduced to the form

$$\sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \Rightarrow x_1\ 2\pi k,\ x_2 = \frac{\pi}{2} + 2\pi k$$

E. SOLVING EQUATIONS BY TRANSFORMING A SUM OF TRIGONOMETRIC FUNCTIONS INTO A PRODUCT

Ex.27 Solve $\cos 3x + \sin 2x - \sin 4x = 0$

Sol. $\cos 3x + \sin 2x - \sin 4x = 0 \Rightarrow \cos 3x + 2\cos 3x \cdot \sin(-x) = 0 \Rightarrow \cos 3x - 2\cos x \cdot \sin x = 0$

$\Rightarrow \cos 3x(1 - 2\sin x) = 0 \Rightarrow \cos 3x = 0$ or $1 - 2\sin x = 0$

$\Rightarrow 3x = (2n+1)\frac{\pi}{2},\ n \in I$ or $\sin x = \frac{1}{2} \Rightarrow x = (2n+1)\frac{\pi}{6},\ n \in I$ or $x = n\pi + (-1)^n\frac{\pi}{6},\ n \in I$

$\therefore$ solution of given equation is $(2n+1)\frac{\pi}{6},\ n \in I$ or $n\pi + (-1)^n\frac{\pi}{6},\ n \in I$

F. SOLVING EQUATIONS BY TRANSFORMING A PRODUCT OF TRIGONOMETRIC FUNCTIONS INTO A SUM

Ex.28 Solve $\sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x$

Sol. $\because \sin 5x \cdot \cos 3x = \sin 6x \cdot \cos 2x \Rightarrow 2\sin 5x \cdot \cos 3x = 2\sin 6x \cdot \cos 2x$

$\Rightarrow \sin 8x + \sin 2x = \sin 8x + \sin 4x \Rightarrow \sin 4x - \sin 2x = 0$

$\Rightarrow 2\sin 2x \cdot \cos 2x - \sin 2x = 0 \Rightarrow \sin 2x(2\cos 2x - 1) = 0$

$\Rightarrow \sin 2x = 0$ or $2\cos 2x - 1 = 0 \Rightarrow 2x = n\pi,\ n \in I$ or $\cos 2x = \frac{1}{2}$

$\Rightarrow x = \frac{n\pi}{2},\ n \in I$ or $2x = 2n\pi \pm \frac{\pi}{3},\ n \in I \Rightarrow x = n\pi \pm \frac{\pi}{6},\ n \in I$

$\therefore$ Solution of given equation is $\frac{n\pi}{2},\ n \in I$ or $n\pi \pm \frac{\pi}{6},\ n \in I$

Ex.29 Solve the equation $\cot x - 2\sin 2x = 1$

Sol. **First solution :** The equation becomes senseless for $x = k\pi$. For all the other values of x it is equivalent to the equation $\cos x - \sin x = 2\sin 2x \cdot \sin x$

we obtain $\cos x - \sin x = \cos x - \cos 3x$, $\sin x = \cos 3x$,

whence $\sin x = \sin\left(\frac{\pi}{2} - 3x\right)$. Consequently, $2\sin\left(2x - \frac{\pi}{4}\right)\cos\left(x - \frac{\pi}{4}\right) = 0 \Rightarrow x_1 = \frac{\pi}{8} + \frac{k\pi}{2}$, $x_2 = \frac{3\pi}{4} + k\pi$

Second solution : putting $\tan x = t$, we get the equation $t^3 + 3t^2 + t - 1 = 0$.

Factoring the left member, we obtain $(t + 1)(t + 1 - \sqrt{2})(t + 1 + \sqrt{2}) = 0$.

whence $(\tan x)_1 = 1$, $(\tan x)_2 = \sqrt{2} - 1$, $(\tan x)_3 = -1 - \sqrt{2}$.

$\Rightarrow$ $x_1 = \frac{3\pi}{4} + k\pi$; $x_2 = \text{arc tan}(\sqrt{2} - 1) + k\pi$, $x_3 = -\text{arc tan}(1 + \sqrt{2}) + k\pi$.

G. SOLVING EQUATIONS BY A CHANGE OF VARIABLE

(i) Equations of the form $P(\sin x \pm \cos x, \sin x \cdot \cos x) = 0$, where $P(y, z)$ is a polynomial, can be solved by the change $\cos x \pm \sin x = t \Rightarrow 1 \pm 2\sin x \cdot \cos x = t^2$.

(ii) Equations of the form of $a \cdot \sin x + b \cdot \cos x + d = 0$, where a, b & d are real numbers & $a, b \neq 0$ can be solved by changing $\sin x$ & $\cos x$ into their corresponding tangent of half the angle.

(iii) Many equations can be solved by introducing a new variable. eg. the equation $\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x$ changes to

$2(y + 1)\left(y - \frac{1}{2}\right) = 0$ by substituting, $\sin 2x \cdot \cos 2x = y$.

Ex.30 Solve sin x + cos x = 1 sin x. cos x

Sol. $\because$ $\sin x + \cos x = 1 \sin x \cdot \cos x$ ...(i) Let $\sin x + \cos x = t$

$\Rightarrow$ $\sin^2 x + \cos^2 x + 2\sin x \cdot \cos x = t^2$ $\Rightarrow$ $\sin x \cdot \cos x = \frac{t^2 - 1}{2}$

Now put $\sin x + \cos x = t$ and $\sin x \cdot \cos x = \frac{t^2 - 1}{2}$ in (i), we get $t = 1 + \frac{t^2 - 1}{2}$

$\Rightarrow$ $t^2 - 2t + 1 = 0$ $\Rightarrow$ $t = 1$ ($\because$ $t = \sin x + \cos x$) $\Rightarrow$ $\sin x + \cos x = 1$ ....(ii)

divide both sides of equation (ii) by $\sqrt{2}$, we get

$\Rightarrow$ $\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$ $\Rightarrow$ $\cos\left(x - \frac{\pi}{4}\right) = \cos\frac{\pi}{4}$ $\Rightarrow$ $x - \frac{\pi}{4} = 2n\pi \pm \frac{\pi}{4}$

(i) if we take positive sign, we get $x = 2n\pi + \frac{\pi}{2}$, $n \in I$

(ii) if we take negative sign, we get $x = 2n\pi$, $n \in I$

Ex.31 Solve the equation $\sin 2x - 12 (\sin x - \cos x) + 12 = 0$

Sol. Putting $\sin x - \cos x = t$ and using the identity $(\sin x - \cos x)^2 = 1 - 2 \sin x \cos x$, we rewrite the original equation in the form $t^2 + 12t - 13 = 0$.

This equation has the roots $t_1 = -13$ and $t_2 = 1$. But $t = \sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$, and thus,

$|t| \le \sqrt{2}$. Consequently, the root $t_1 = -13$ must be discarded. Therefore, the original equation is

reduced to the equation $\sin\left(x - \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$. $\Rightarrow$ $x_1 = \pi + 2k\pi$, $x_2 = \frac{\pi}{2} + 2k\pi$.

Ex.32 Solve the equation $1 + 2 \csc x = -\dfrac{\sec^2 \frac{x}{2}}{2}$.

Sol. Transform the given equation to the form $2 \cos^2 \frac{x}{2} (2 + \sin x) + \sin x = 0$.

Using the formula $2 \cos^2 \frac{x}{2} = 1 + \cos x$ and opening the brackets, we obtain

$2 + 2 (\sin x + \cos x) + \sin x . \cos x = 0$. ....(1)

By the substitution $\sin x + \cos x = t$ equation (1) is reduced to the quadratic equation $t^2 + 4t + 3 = 0$ whose roots are $t_1 = -1$ and $t_2 = -3$. Since $|\sin x + \cos x| \le \sqrt{2}$, the original equation can only be satisfied by the roots of the equation $\sin x + \cos x = -1$. ....(2)

Solving equation (2), we obtain $x_1 = -\frac{\pi}{2} + 2k\pi$ and $x_2 = (2k + 1)\pi$.

here x_2 should be discarded because $\sin x_2 = 0$, and therefore the original equation makes no sense for

$x = x_2$ $\Rightarrow$ $x = -\frac{\pi}{2} + 2k\pi$

Ex.33 Solve the equation $\sin x + \cos x - 2\sqrt{2} \sin x \cos x = 0$.

Sol. Designating $\sin x + \cos x = t$ and using the equation $\sin x \cos x = (t^2 - 1)/2$, we reduce the equation to a new equation with respect to t : $\sqrt{2}t^2 - t - \sqrt{2} = 0$.

The numbers $t_1 = \sqrt{2}$, $t_2 = -\frac{1}{\sqrt{2}}$ are roots of this quadratic equation.

Thus the solution of the initial equation reduces to the solution of the trigonometric equations :

$\sin x + \cos x = \sqrt{2}$, $\sin x + \cos x = -\frac{1}{\sqrt{2}}$.

Multiplying both sides of these equations by the number $\frac{1}{\sqrt{2}}$, we reduce them to two simpler equations:

$$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = 1 \Leftrightarrow \sin x \cos \frac{\pi}{4} + \sin \frac{\pi}{4} \cos x = 1 \Leftrightarrow \sin\left(x + \frac{\pi}{4}\right) = 1.$$

$$\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x = -\frac{1}{2} \Leftrightarrow \sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{2}.$$

The solutions of the equations $\sin\left(x + \frac{\pi}{4}\right) = 1$ and $\sin\left(x + \frac{\pi}{4}\right) = -\frac{1}{2}$ are

$$x = \frac{\pi}{4} + 2\pi k, \quad k \in Z; \qquad x = (-1)^{n+1} \frac{\pi}{6} - \frac{\pi}{4} + \pi n, \quad n \in Z.$$

Ex.34 Solve the equation $\frac{1}{2}(\sin^4 x + \cos^4 x) = \sin^2 x \cos^2 x + \sin x \cos x$

Sol. We obtain the equation $\sin^2 2x + \sin 2x - 1 = 0$

Solving it, we get $\sin 2x = \frac{\sqrt{5}-1}{2} \Rightarrow x = (-1)^k \frac{1}{2} \text{arc} \sin \frac{\sqrt{5}-1}{2} + \frac{k\pi}{2}$

Ex.35 Find $\tan \frac{\alpha}{2}$ if $\sin \alpha + \cos \alpha = \frac{\sqrt{7}}{2}$ and the angle α lies between 0° and 45°.

Sol. Using formulas, we reduce the given relation $\sin \alpha + \cos \alpha = \frac{\sqrt{7}}{2}$ to the form

$$(2 + \sqrt{7}) \tan^2 \frac{\alpha}{2} - 4 \tan \frac{\alpha}{2} - (2 - \sqrt{7}) = 0$$

Solving this equation with respect to $\tan \frac{\alpha}{2}$, we obtain $\left(\tan\frac{\alpha}{2}\right)_1 = \frac{3}{2+\sqrt{7}} = \sqrt{7} - 2$ & $\left(\tan\frac{\alpha}{2}\right)_2 = \frac{\sqrt{7}-2}{3}$

Let us verify whether the above values of $\tan \frac{\alpha}{2}$ satisfy the conditions of the problem.

Since $0 < \frac{\alpha}{2} < \frac{\pi}{8}$, we have the condition $0 < \tan \frac{\alpha}{2} < \tan \frac{\pi}{8} = \sqrt{2} - 1$.

The value $\left(\tan\frac{\alpha}{2}\right)_2 = \frac{\sqrt{7}-2}{3}$ satisfies this condition because $\frac{\sqrt{7}-2}{3} < \sqrt{2}-1$. The root $\sqrt{7}-2$ should be discarded since $\sqrt{7}-2 > \sqrt{2}-1$.

Ex.36 Solve 3 cos x + 4 sin x = 5

Sol. $\because$ 3 cos x + 4 sin x = 5 $\because$ cos x = $\frac{1-\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}}$ & sin x = $\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$\therefore$ equation (i) becomes $\Rightarrow$ $3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$ = 5 ....(ii)

Let tan $\frac{x}{2}$ = t $\therefore$ equation (i) becomes $3\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)$ = 5 $\Rightarrow$ $4t^2 - 4t + 1 = 0$

$\Rightarrow$ $(2t-1)^2 = 0 \Rightarrow t = \frac{1}{2}$ ($\because$ t = tan $\frac{x}{2}$) $\Rightarrow$ tan $\frac{x}{2} = \frac{1}{2}$ $\Rightarrow$ tan $\frac{x}{2}$ = tan α, where tan α = $\frac{1}{2}$

$\Rightarrow$ tan $\frac{x}{2}$ = $n\pi + a$ $\Rightarrow$ $x = 2n\pi + 2\alpha$ where $\alpha = \tan^{-1}\left(\frac{1}{2}\right)$, $n \in I$

Ex.37 Solve the equation $2 + \cos x = 2\tan\frac{x}{2}$

Sol. Write the equation in the following form : $\cos^2\frac{x}{2} - \sin^2\frac{x}{2} = 2\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}-1\right)$

After some simple transformations it is reduced to the equation

$$\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)\left(3\cos^2\frac{x}{2}+2\sin^2\frac{x}{2}+\sin\frac{x}{2}\cos\frac{x}{2}\right) = 0.$$

The equation $3\cos\frac{x}{2} + 2\sin^2\frac{x}{2} + \sin\frac{x}{2}\cos\frac{x}{2} = 0$ is equivalent to the equation

$2\tan^2\frac{x}{2} + \tan\frac{x}{2} + 3 = 0$ and has no real solutions. $x = \frac{\pi}{2} + 2k\pi$.

H. SOLVING EQUATIONS WITH THE USE OF THE BOUNDNESS OF THE FUNCTIONS SIN *X* & COS *X*

Ex.38 Solve the equation $\frac{1-\tan x}{1+\tan x}$ = 1 + sin 2x.

Sol. The equation makes no sense for x = $\frac{\pi}{2}$ + kπ and for x = $-\frac{\pi}{4}$ + kπ. For all the other values of x it is equivalent to the equation $\frac{\cos x-\sin x}{\cos x+\sin x}$ = 1 + sin 2x.

After simple transformations we obtain sin x (3 + sin 2x + cos 2x) = 0.

It is obvious that the equation sin 2x + cos 2x + 3 = 0 has no solution, and therefore, the original equation is reduced to the equation sin x = 0 ⇒ x = kπ

Ex.39 Solve the equation (sin x + cos x) $\sqrt{2}$ = tan x + cot x.

Sol. Let us transform the equation to the form $\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{2\sin x\cos x}$ or $\sin\left(x+\frac{\pi}{4}\right)=\frac{1}{\sin 2x}$,

i.e. $\sin\left(x+\frac{\pi}{4}\right)$ sin 2x = 1. ....(1)

We have |sin α| ≤ 1, and therefore (1) holds

if either $\sin\left(x+\frac{\pi}{4}\right)$ =1 and sin 2x = –1 or $\sin\left(x+\frac{\pi}{4}\right)$ = 1 and sin 2x = 1.

But the first two equations have no roots in common while the second two equations have the common roots x = $\frac{\pi}{4}$ + 2kπ. Consequently the roots of the given equation are x = $\frac{\pi}{4}$ + 2kπ.

Ex.40 Solve the equation $\sin^{2n-1}x + 2\cos^{2n-1}x = 2$, where n ∈ N.

Sol. Obviously no solution is possible if $\frac{\pi}{2}$ < x < 2π as LHS < 2.

If 0 < x < $\frac{\pi}{2}$, then LHS = $\sin^{2n-1}x + 2\cos^{2n-1}x < \sin^2 x + 2\cos^2 x = 1 + \cos^2 x < 2$ when n ∈ N – {1}.

Obviously, a solution exists only when x = 0 ⇒ The general solution is x = 2mπ, m ∈ I.

When n = 1 sin x + 2 cos x = 2 $\sin\frac{x}{2}\left(2\sin\frac{x}{2}-\cos\frac{x}{2}\right)$ = 0

⇒ either x = $2k_1\pi$ or x = $2k_2\pi + 2\tan^{-1}\frac{1}{2}$, k_1, k_2 ∈ I.

Ex.41 Find all possible real values of x and y satisfying $\sin^2 x + 4\sin^2 y - \sin x - 2\sin y - 2\sin x.\sin y + 1 = 0$.

Sol. Given equation can be rewritten as $\sin^2 x - \sin x\,(1 + 2)\,(1 + 2\sin y) + 4\sin^2 y - 2\sin y + 1 = 0$

$$\Rightarrow \quad \sin x \; \frac{(1+2\sin y) \pm \sqrt{(1+2\sin y)^2 - 4(4\sin^2 y - 2\text{siy} + 1)}}{2}$$

$$\Rightarrow \quad \sin x = \frac{(1+2\sin y) \pm \sqrt{-3 - 12\sin^2 y + 12\sin y}}{2} = \frac{(1+2\sin y) \pm \sqrt{-3(2\sin y - 1)^2}}{2}$$

Since sinx is real, $\Rightarrow$ $2\sin y - 1 = 0$ or $\sin y = 1/2$ and $\sin x = \frac{1+1}{2} = 1$

$\Rightarrow$ $y = n_1\pi + (-1)^n \frac{\pi}{6}$, $x = (4n_2 + 1)\frac{\pi}{2}$ where $n_1, n_2 \in I$.

Ex.42 Solve the equation $\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] - \tan^2\left(x + \frac{\pi}{4}\tan^2 x\right) = 1$.

Sol. $\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] - \tan^2\left(x + \frac{\pi}{4}\tan^2 x\right) = 1$

since square of the cosine of any argument doesn't exceed 1, the given equation holds true if and only if we have, simultaneously $\cos^2\left[\frac{\pi}{4}(\sin x + \sqrt{2}\cos^2 x)\right] = 1$ ...(1)

and $\tan\left(x + \frac{\pi}{4}\tan^2 x\right) = 0$ ...(2) from (1), $\sin x + \sqrt{2}\cos^2 x = 4k$ ...(3) $\forall\, k \in I$

but $|\sin x + \sqrt{2}\cos^2 x| \le |\sin x| + \sqrt{2}\,|\cos^2 x| \le 1 + \sqrt{2} < 4$

so, equation (3) has no solution for $k \ne 0$ for $k = 0$

$\sin x + \sqrt{2}\cos^2 x = 0$ or, $\sqrt{2}\sin^2 x - \sin x - \sqrt{2} = 0$ or, $\sin x = \frac{-1}{\sqrt{2}}, \sqrt{2}$

but $\sin x = \sqrt{2}$ is not possible. so only solution to the equation (1) is

$x_1 = \frac{-\pi}{4} + 2n\pi_1$ $x_2 = \frac{5\pi}{4} + 2n\pi_1 n = 0, \pm1, \pm2$.........

for $x_1 = \frac{-\pi}{4} + 2n\pi$, equation (2) becomes an identity but $x_2 = \frac{5\pi}{4} + 2n\pi$ doesn't satisfy equation (2)

so, solution to the original equation $x = \frac{-\pi}{4} + 2n\pi\ \forall\, n \in I$

Ex.43 Find the general solution of the equation, $\sin 3x + \cos 4x - 4\sin 7x = \cos 10x + \sin 17x$.

Sol. $(\sin 17x - \sin 3x) - \cos 10x - \cos 4x + 4\sin 7x = 0 \Rightarrow 2\cos 10x\ \sin 7x + 2\sin 7x\ \sin 3x + 4\sin 7x = 0$

$\Rightarrow$ $\sin 7x\,(\cos 10x - \sin 3x + 2) = 0$ Hence $\sin 7x = 0 \Rightarrow x = \frac{n\pi}{7}$, $n \in I$

or $\cos 10x - \sin 3x + 2 = 0$ $\Rightarrow$ $\cos 10x = -1$ and $\sin 3x = 1$ given $x = (4n + 1)\frac{\pi}{6}$

i.e. $x = -\frac{3\pi}{6}^*$, $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{9\pi}{6}^*$, $\frac{13\pi}{6}$, $\frac{17\pi}{6}$, $\frac{21\pi}{6}^*$, ... , $\frac{33\pi}{6}$ ......

Those starred also satisfy $\cos 10x = -1$, the general term of which is

$x = 3(4k-1)\frac{\pi}{6}$ $k \in I$ Hence $x = \frac{n\pi}{7}$ or $3(4k-1)\frac{\pi}{6}$ where $n, k \in I$

I. SIMULTANEOUS EQUATIONS

Ex.44 Solve the system of equations form $\left.\begin{array}{l}\sin x = \csc x + \sin y,\\ \cos x = \sec x + \cos y.\end{array}\right\}$

Sol. Transform the system to the $\left.\begin{array}{l}\sin^2 x = 1 + \sin x \sin y,\\ \cos^2 x = 1 + \cos x \cos y.\end{array}\right\}$ ....(1)

Adding together the equations of system (1) and subtracting the first equation form the second we obtain the system $\left.\begin{array}{l}\cos 2x - \cos(x = y) = 0,\\ 1 + \cos(x - y) = 0.\end{array}\right\}$ ....(2)

The first equation of system (2) can be rewritten as $\cos 2x - \cos(x+y) = 2\sin\left(\frac{3x+y}{2}\right)\sin(y-x) = 0$.

If $\sin(x - y) = 0$, then $x - y = k\pi$. But from the second equation of system (2) we find
$\cos(x - y) = -1$, $x - y = (2n + 1)\pi$.
Consequently, in this case we have an infinitude of solutions : $x - y\ (2n + 1)\pi$.

If $\sin\left(\frac{3x+y}{2}\right) = 0$, then $3x = y = 2k\pi$. But $x - y = (2n + 1)\pi \Rightarrow x = \frac{2k+2n+1}{4}\pi$, $y = \frac{2k-6n-3}{4}\pi$

Ex.45 Solve the system of equations $\sin x \sin y = \frac{\sqrt{3}}{4}$, $\cos x \cos y = \frac{\sqrt{3}}{4}$.

Sol. Adding up the equations of the system, we arrive at an equation

$\sin x \sin y + \cos x \cos y = \frac{\sqrt{3}}{2} \Leftrightarrow \cos(x - y) = \frac{\sqrt{3}}{2}$.

Subtracting the first equation of the system from the second. we arrive at an equation
$\cos x \cos y - \sin x \sin y = 0 \Leftrightarrow \cos(x + y) = 0$,

Thus the initial system is equivalent to the system $\cos(x - y)\ \frac{\sqrt{3}}{2}$, $x - y = \pm\frac{\pi}{6} + 2\pi n$,

$\Leftrightarrow n, k \in Z$, $\cos(x + y) = 0$, $x + y = \frac{\pi}{2} + \pi k$, whence $x = \frac{\pi}{3} + \frac{\pi}{2}(2n + k)$, $x = \frac{\pi}{6} + \frac{\pi}{2}(2n + k)$,

$y = \frac{\pi}{6} + \frac{\pi}{2}(k - 2n)$, $y = \frac{\pi}{3} + \frac{\pi}{2}(k - 2n)$.

$\Rightarrow \frac{\pi}{3} + \frac{\pi}{2}(2n - k)$, $\frac{\pi}{6} + \frac{\pi}{2}(k - 2n)$; $\frac{\pi}{6} + \frac{\pi}{2}(2n + k)$, $\frac{\pi}{3} + \frac{\pi}{2}(k - 2n)$ $(k, n \in Z)$.

J. MISCELLANEOUS QUESTIONS

Ex.46 Solve the equation 2 cot 2x – 3 cot 3x = tan 2x.

Sol. The give equation can be rewritten in the form $3\left(\frac{\cos 2x}{\sin 2x}-\frac{\cos 3x}{\sin 3x}\right)=\frac{\sin 2x}{\cos 2x}+\frac{\cos 2x}{\sin 2x}$

or $\frac{3\sin x}{\sin 2x \sin 3x}=\frac{1}{\sin 2x \cos 2x}$.

Note that this equation has sense if the condition sin 2x ≠ 0, sin 3x ≠ 0, cos 2x ≠ 0 holds. For the values of x satisfying this condition we have 3 sin x cos 2x = sin 3x. Transforming the last equation we obtain sin x (3 – 4 $\sin^2$ x – 3 cos 2x) = 0 and thus arrive at the equation 2 $\sin^3$ x = 0, which is equivalent to the equation sin x = 0. Hence, due to the above note, the original equation has no solutions.

Ex.47 Solve the equation $\tan\left(x-\frac{\pi}{4}\right)\tan x \tan\left(x+\frac{\pi}{4}\right)=\frac{4\cos^2 x}{\tan\frac{x}{2}-\cot\frac{x}{2}}$

Sol. The right-hand side of the equation is not determined for x = kπ and x = π/2 + mπ, because for x = 2/π the function cot x/2 is not defined, for x = (2/π + 1) π the function tan x/2 is not defined and for x = π/2 + mπ the denominator of the right member of the right member vanishes. For x ≠ kπ we have

$$\tan\frac{x}{2}-\cot\frac{x}{2}=\frac{\sin^2\frac{x}{2}-\cos^2\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}=-\frac{2\cos x}{\sin x}.$$

Hence, for x ≠ kπ and x ≠ $\frac{\pi}{2}$ + mπ (where k and m are arbitrary integers) the right member of the equation is equal to – 2 sin x cos x.

The left member of the equation has no sense for x = $\frac{\pi}{2}$ + kπ and x = $\frac{\pi}{4}$ + ℓ . $\frac{\pi}{2}$ (ℓ=0, ±1, ±2, ...), and for all the other values of x it is equal to –tan x because

$$\tan\left(x-\frac{\pi}{4}\right)\tan\left(x+\frac{\pi}{4}\right)=\tan\left(x-\frac{\pi}{4}\right)\cot\left[\frac{\pi}{2}-\left(x+\frac{\pi}{4}\right)\right]=-\tan\left(x-\frac{\pi}{4}\right)\cot\left(x-\frac{\pi}{4}\right)=-1.$$

Thus, if x ≠ kπ, x ≠ $\frac{\pi}{2}$ + mπ and x ≠ $\frac{\pi}{4}$ + l $\frac{\pi}{2}$, then the original equation is reduced to the form tan x = 2 sin x cos x.

This equation has the roots x = kπ and x = $\frac{\pi}{4}+\ell\frac{\pi}{2}$ It follows that the original equation has no roots.

Ex.48 Solve the equation 8 $\sin^6$ x + 3 cos 2x + 2 cos 4x + 1 = 0

Sol. Applying the formulas $\sin^2\alpha=\frac{1-\cos 2\alpha}{2}$ and cos 2α = 2 $\cos^2$ α – 1

we rewrite the equation in the form $(1 - \cos 2x)^3$ + 3cos 2x + 2(2 $\cos^2$ 2x – 1) + 1 = 0,

or 7 $\cos^2$ 2x – $\cos^3$ 2x = 0, whence cos 2x = 0, x = $\frac{\pi}{4}$ + k $\frac{\pi}{2}$.

Ex.49 Solve the equation $(1 + k) \cos x \cos(2x - \alpha) = (1 + k \cos 2x) \cos (x - \alpha)$

Sol. Let us rewrite the given equation in the form

$(1 + k) \cos x \cos (2x - \alpha) = \cos (x - \alpha) + k \cos 2x \cos (x - \alpha)$. ....(1)

We have $\cos x \cos (2x - \alpha) = \frac{1}{2} [\cos (3x - \alpha) + \cos (x - \alpha)]$

and $\cos (x - \alpha) \cos 2x = \frac{1}{2} [\cos (3x - \alpha) + \cos (x + \alpha)]$,

and therefore equation (1) turns into $k [\cos (x - \alpha) - \cos (x + \alpha)] = \cos (x - \alpha) - \cos (3x - \alpha)$, that is $k \sin x \sin \alpha = \sin (2x - \alpha) \sin x$. ....(2)

Equation (2) is equivalent to the following two equations ;

(a) $\sin x = 0$; $x = l\pi$ and (b) $\sin (2x - \alpha) = k \sin \alpha$.

Thus, $x = \frac{\alpha}{2} + (-1)^n , \frac{1}{2} \text{arc} \sin (k \sin \alpha) + \frac{\pi}{2} n$.

For the last expression to make sense, k and α must satisfy the condition $| k \sin \alpha| \le 1$.

Ex.50 Find all solutions of the equation $1 + (\sin x - \cos x) \sin \frac{\pi}{4} = 2 \cos^2 \frac{5}{2} x$, ...(1)

which satisfy the condition $\sin 6x < 0$. ....(2)

Sol. Let us simplify the initial equation : $1 + (\sin x - \cos x) \sin \frac{\pi}{4}$

$= 2 \cos^2 \frac{5x}{2} \Leftrightarrow 1 + (\sin x - \cos x) \frac{\sqrt{2}}{2} = 1 + \cos 5x \Leftrightarrow \cos 5x + \cos \left(x + \frac{\pi}{4}\right) = 0$,

$2 \cos \left(3x + \frac{\pi}{8}\right) \cos \left(2x - \frac{\pi}{8}\right) = 0$.

Thus initial equation (1) is equivalent to the equations $\cos \left(3x + \frac{\pi}{8}\right) = 0$, $\cos \left(2x - \frac{\pi}{8}\right) = 0$, ...(3)

whose roots are equal, respectively, to $x = \frac{\pi}{8} + \frac{\pi n}{3}$, $n \in Z$, $x = \frac{5\pi}{16} + \frac{\pi n}{2}$, $n \in Z$.

The least common multiple of the periods of the trigonometric functions entering into equation (1) and inequality (2) is equal to 2π. From the obtained solutions of the equation belonging to the interval $[0, 2\pi)$ the numbers $\frac{5\pi}{16}$ and $\frac{5\pi}{16} + \pi$ satisfy inequality (2). All the solutions of the problem can be obtained by adding number, which are multiples of 2π, to each root obtained $x = \frac{5\pi}{16} + \pi k$ $(k \in Z)$

Ex.51 Solve the equation (cos x – sin x) $\left(2\tan x + \frac{1}{\cos x}\right)$ + 2 = 0

Sol. We designate t = tan $\frac{x}{2}$ and, using the formulas of the universal trigonometric substitution, write the equation in the form $\frac{3t^4+6t^3+8t^2-2t-3}{(t^2+1)(1-t^2)}$ = 0, its roots are $t_1 = \frac{1}{\sqrt{3}}$, $t_2 = -\frac{1}{\sqrt{3}}$. Thus the solution of the equation reduces that of two elementary equations $\tan\frac{x}{2} = \frac{1}{\sqrt{3}}$, $\tan\frac{x}{2} - \frac{1}{\sqrt{3}}$. ....(1)

Verification shows that the numbers πn which are roots of the equation cos $\frac{\pi}{2}$ = 0, are not the roots of the given equation, and consequently, all solutions of the initial equation can be found as solutions of equation (1) $\Rightarrow$ $x = \pm\frac{\pi}{3} + 2\pi k$ $(k \in Z)$.

Ex.52 Solve the equation, $5\sin x + \frac{5}{2\sin x} - 5 = 2\sin^2 x + \frac{1}{2\sin^2 x}$ if $x \in (0, \pi)$.

Sol. $5\left(\sin x + \frac{1}{2\sin x}\right) - 5 = 2\left(\sin^2 x + \frac{1}{4\sin^2 x}\right) = 2\left(\left(\sin x + \frac{1}{2\sin x}\right)^2 - 1\right)$

Let $\sin x + \frac{1}{2\sin x} = t$ $\Rightarrow$ $5t - 5 = 2(t^2 - 1)$ $\Rightarrow$ $2t^2 - 5t + 3 = 0$ $\Rightarrow$ $(2t - 3)(t - 1) = 0$

$\Rightarrow$ t = 1 or t = 3/2

If t = 1, $2\sin^2 x - 2\sin x + 1 = 0$ D < 0 no solution

If t = 3/2, $2\sin^2 x - 3\sin x + 1 = 0$ $\Rightarrow$ sin x = 1 or sin = 1/2

$\therefore$ $x = \frac{\pi}{2}$ or $\frac{\pi}{6}, \frac{5\pi}{6}$ $\Rightarrow$ $x \in \left\{\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}\right\}$

Ex.53 Solve |sin 3x + sin x| + |sin 3x – sin x| = $\sqrt{3}. -\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

Sol. |sin 3x + sin x| = |2 sin 3x cos x| = sin 3x + sin x , $0 \le x < \frac{\pi}{2}$

= – sin 3x – sin x , $-\frac{\pi}{2} < x < 0$

|sin 3x – sin x| = |2 cos 2x sin x| = sin 3x – sin x , $\left(0 \le x \le \frac{\pi}{4}\right) \cup \left(-\frac{\pi}{2} < x < -\frac{\pi}{4}\right)$

= – sin 3x + sin x , $\left(\frac{\pi}{4} < x < \frac{\pi}{2}\right) \cup \left(-\frac{\pi}{4} < x < 0\right)$

$\Rightarrow$ $|\sin 3x + \sin x| + |\sin 3x - \sin x| = 2 \sin 3x = \sqrt{3}$ $0 \le x \le \frac{\pi}{4} \Rightarrow 3x = np + (-1)^n \frac{\pi}{3}$, $x = \frac{\pi}{9}, \frac{2\pi}{9}$

$= |\sin 3x + \sin x| + |\sin 3x - \sin x| = 2 \sin x = \sqrt{3}$ $\frac{\pi}{4} < x \frac{\pi}{2} \Rightarrow$ $x = n\pi + (-1)^n \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3}$

As $|\sin 3x + \sin x| + |\sin 3x - \sin x|$ is even function hence – $-\frac{2\pi}{9}, -\frac{\pi}{9}, -\frac{\pi}{3}$ will also satisfy it.

Ex.54 Find all the values of a for which every real root of the equation $\cos 3x = a \cos x + (4 - 2|a|) \cos^2 x$ is a root of the equation $\cos 3x + \cos 2x = 2 \cos x \cos 2x - 1$ and vice versa.

Sol. Put $\cos x = y$

We get, $4y^3 - 3y = ay + 2(2 - |a|) y^2 \Rightarrow 4y^3 - 2(2 - |a|) y^2 - (a + 3) y = 0$ ....(1)

& $4y^3 - 3y + 2y^2 - 1 = 2y(2y^2 - 1) - 1 \Rightarrow 4y^3 - 3y + 2y^2 - 1 = 4y^3 - 2y - 1 \Rightarrow 2y^2 - y = 0$ ....(2)

$y = 0$ and $\frac{1}{2}$ are the roots of the equation (2)

Now, we have to find the values of a for which equation (1) have the roots either $y = 0$, $y = \frac{1}{2}$

This is possible only when third root of the equation (1) is either $y = 0$, $\frac{1}{2}$ or $|y| > 1$

($\because$ $|\cos x| = |y| > 1$) clearly $y = 0$ is a solution of equation (1)

Now we will henceforth consider the equation $4y^2 - 2(2 - |a|) y - (a + 3) = 0$

One of the root of this equation must be $\frac{1}{2}$.

Substituting $y = \frac{1}{2}$ in to it, we find that $\frac{1}{2}$ is a root, when $|a| = a + 4$. Now for this value of a the other root of the equation will be $y = -\frac{(a+3)}{2}$.

Now, value of a will be suitable in the following three cases :

Case I : $\frac{a+3}{2} = 0$, Case II : $\frac{a+3}{2} = \frac{1}{2}$, Case III : $\left|-\frac{a+3}{2}\right| > 1 \Rightarrow a = -3, a = -4, a < -5, -1 < a < 0$.

Ex.55 Let A and B be acute positive angles satisfying the equalities $3 \sin^2 A + 2 \sin^2 B = 1$;

$3 \sin 2A - 2 \sin 2B = 0$. Prove that $A + 2B = \frac{\pi}{2}$.

Sol. From the given relations we get $\sin 2B = \frac{3}{2} \sin 2A$, $3 \sin^2 A = 1 - 2 \sin^2 B = \cos 2B$,

hence $\cos (A + 2B) = \cos A \cos 2B - \sin A \sin 2B = \cos A \,.\, 3 \sin^2 A - \frac{3}{2} \sin A \sin 2A = 0$.